International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

Unitary Self-Adjoint Logics of Projections

Marjan Matvejchuk[1]

Received July 4, 1997

Quantum logics of projections being self-adjoint with respect to a unitary operator on a Hilbert space are studied.

1. INTRODUCTION

In Matvejchuk (1995), a universal method for the construction of quantum logics of projections was given. Let H be a Hilbert space with the inner product (.,.). Let A be a linear or conjugate linear invertible bounded operator in H. Put $\langle x,y \rangle = (Ax, y)$, $\forall x, y \in H$. It is clear that $B \in B(H)$ is A-self-adjoint, i.e., $\langle Bx,y \rangle = \langle x, By \rangle$, $\forall x, y \in H$ iff $B = A^{-1}B^*A$. Denote by P_A the set $\{p \in B(H)$: $p^2 = p$, $\langle px, y \rangle = \langle x, py \rangle$, $\forall x, y \in H\}$ of all A-self-adjoint projections (=idempotents). Let Π_A be the set of all orthogonal projections from P_A. With respect to the standard relations, the ordering $p \le g$ iff $pg = gp = p$ and the orthocomplementation $p \to p^\perp \equiv I - p$ the set P_A is a quantum logic.

For the case $A > 0$, the logic P_A is isomorphic to the lattice P_I of all orthogonal projections in $B(H)$ (see Matvejchuk, 1989). In Matvejchuk (1995, n.d.), the hyperbolic logic P_J and the conjugation logic $P_{\mathcal{T}}$ was studied, where J is a symmetry ($J^* = J$, $J^2 = I$) and $\mathcal{T}$ is an operator conjugation.

In this paper, a logic P_U, where U is an unitary operator, is studied.

2. THE STRUCTURE OF THE PROJECTIONS IN P_U

Proposition 1. If $p \in P_U$, then $p^* \in P_U$, $pU = Up^*$, $Up = p^*U$ and $pU^2 = U^2p$.

[1] Department of Physics and Mathematics, Pedagogical University, 432700 Uljanovsk, Russia; e-mail: Marjan.Matvejchuk@ksu.ru.

0020-7748/98/0100-010 3$15.00/0

Proof. Let $p \in P_U$. Then $p = U^*p^*U$. This means that $p^* = UpU^*$ and $p^* = (U^*p^*U)^* = U^*pU$. Hence $p^* \in P_U$ and $Up = p^*U$, $Up^* = pU$. In addition, $UpU^{-1} = p^* = U^{-1}pU$ implies $U^2p = pU^2$.

Proposition 2. Let p be a bounded projection. Denote by e the orthogonal projection onto $pH \cap p^*H$. Then e is the greatest orthogonal projection with the property $e \leq p$. If $p \in P_U$, then $e \in \Pi_U$.

Proof. It is clear that $ep = (p^*e)^* = e^* = e = pe$. Thus $e \leq p$. Assume that there exists an orthogonal projection r such that $r \leq p$. Then $r \leq p^*$. Therefore, $rH \subseteq pH \cap p^*H$. This means that $r \leq e$.

Now, assume $p \in P_U$. Let $y \in pH \cap p^*H$, and let $x, x_0 \in H$ be such that $y = px$ and $y = p^*x_0$. By Proposition 1,

$$pU^{-1}x_0 = U^{-1}p^*x_0 = U^{-1}y = U^{-1}px = U^{-2}p^*Ux = p^*U^{-1}x$$

and $pUx_0 = Up^*x_0 = Uy = Upx = p^*Ux$. Thus Uy, $U^{-1}y \in pH \cap p^*H$. This means that $U^{-1}eU \geq e$ and $UeU^{-1} \geq e$. Hence $U^{-1}eU = e \in \Pi_U$.

Definition 3. The projection e from Proposition 2 is said to be *the orthogonal component* of p and is denoted by p_{or}. A projection $p \in P_U$ is said to be a *properly skew projection* if $p_{or} = 0$.

It is clear that for any $p \in P_U$, $p \neq p^*$ the projection $p - p_{or}$, is a properly skew projection.

The following proposition was proved in Matvejchuk (1998).

Proposition 4. Let p be a bounded projection. Denote by $(p + p^*)_+$ the positive part of the $p + p^*$. Let e_+ be the orthogonal projection onto $(p + p^*)_+H$.. Then $e_+pe_+ = \frac{1}{2}(p + p^*)_+$ and $e_+pe_+ \geq e_+$.

Denote by $\mathcal{A}_U$ the von Neumann algebra $\{a \in B(H)\colon aU = Ua\}$.

Remark 5. If $p \in P_U$, then $e_+pe_+ \in \mathcal{A}_U$.

Proof. Let $p + p^* = \int \lambda\, de_\lambda$ be the spectral decomposition for $p + p^*$. We have $U^{-1}(p + p^*)U = p^* + p$. Hence $p + p^* \in \mathcal{A}_U$. By the uniqueness of the spectral decomposition, $e_\lambda \in \mathcal{A}_U$, $\forall\lambda$. Hence, $e_+ \in \mathcal{A}_U$, too. Finally, $e_+pe_+ = \frac{1}{2}e_+ (p + p^*)e_+ \in \mathcal{A}_U$.

Put $e_- \equiv I - e_+$. Denote by F_y the orthogonal projection onto $\overline{yH}$, $\forall y \in B(H)$.

In Matvejchuk (1995, 1998), there are hyperbolic and conjugation analogies of the following proposition with similar proofs.

Proposition 6. Let $p \in P_U$, and let $e_-pe_+ = w|e_-pe_+|$ be the polar decomposition for e_-pe_+. Then $x \equiv e_+pe_+ (\geq e_+) \in \mathcal{A}_U$, and $U^{-1}wU = -w$, and the formula

$$p = x + w(x^2 - x)^{1/2} - (x^2 - x)^{1/2}w^* - w(x - F_x)w^* \qquad (1)$$

holds.

Conversely, let $x \in \mathcal{A}_U$ be such that $x \geq F_x$, and let there be an isometry w, $U^{-1}wU = -w$, with the initial projection F_x and a final projection e such that $e \perp F_x$. Then (1) defines a projection from P_U.

Proof. Let $p \in P$. By Remark 6, $x \in \mathcal{A}_U$. We have

$$\begin{aligned} e_+pe_- &= \tfrac{1}{2}\, e_+(p + p^*)e_- + \tfrac{1}{2}\,(e_+pe_- - e_+p^*e_-) \\ &= 0 + \tfrac{1}{2}\,(e_+pe_- - e_+p^*e_-) \end{aligned}$$

Hence

$$e_+pe_- = -e_+p^*e_- = -(e_-pe_+)^* \qquad (2)$$

Similarly,

$$e_-pe_+ = -e_-p^*e_+ = -(e_+pe_-)^*$$

By (2), we have

$$\begin{aligned} |e_-pe_+| &= ((e_-pe_+)^*(e_-pe_+))^{1/2} = (-(e_+pe_-)e_-pe_+)^{1/2} \\ &= (e_+p(e_+ - I)pe_+)^{1/2} = ((e_+pe_+)(e_+pe_+) - e_+pe_+)^{1/2} \\ &= (x^2 - x)^{1/2} \end{aligned}$$

Thus $e - pe_+ = w\,(x^2 - x)^{\frac{1}{2}}$ and $e_+pe_- = -(e_-pe_+)^* = -(x^2 - x)^{\frac{1}{2}} w^*$. It is clear that $x|e_-be_+| = |e_-pe_+|x$ and

$$\begin{aligned} U^{-1}wU|epe_+| &= U^{-1}w|e_-pe_+|U = U^{-1}(e_-pe_+)U = e_-U^{-1}pUe_+ \\ &= e_-p^*e_+ = -e_-pe_+ = -w|e_-pe_+| \end{aligned}$$

Hence $U^{-1}wU = -w$.

Now, we show the equality $e_-pe_- = -w\,(x - F_x)w^*$. We have

$$e_+pe_- = (e_+pe_+)(e_+pe_-) + (e_+pe_-)(e_-pe_-)$$

i.e., $(x^2 - x)^{1/2}\, w^* = x(x^2 - x^{1/2}\, w^* + (x^2 - x)^{1/2}\, w^*(e_-pe_-)$. Hence

$$(x^2 - x)^{1/2}(F_x - x)w^* = (x^2 - x)^{1/2}w^*(e_-be_-) \qquad (3)$$

If $z \in e_-H \ominus \overline{e_-pH}$, then $w^*z = 0$ and $e_-pe_-z = 0$. This means that $-w\,(x - F_x)w^*z = e_-pe_-z$. If $z \in e_-pH$, then $w^*z \in (x^2 - x)^{1/2} H$. By (3), $(F_x - x)w^*z = w^*(e_-pe_-)z$, i.e, $-w\,(x - F_x)w^*z = e_-p_-z$, $\forall z \in H$. The proof of (1) is completed.

Let $x \in \mathcal{A}_U$ be such that $x \geq F_x$, and let w, $U^{-1}wU = -w$, be an arbitrary isometry with the initial projection F_x and a final projection e, where

$e \perp F_x$. Note that $U^{-1}wU = -w$ implies $U^{-1}w^*U = -w^*$. Hence $U^{-1}ww^*U = ww^*$ and $U^{-1}w^*wU = w^*w$. Thus $e = w^*w \in \mathcal{A}_U$. Using the right-hand side of (1), define p. It can be easily verified that $p^2 = p$ and $U^{-1}p^*U = p$. Hence $p \in P_U$. QED

To describe measures on a projection logic, it turns out to be useful to know its one-dimensional projections.

Now we give an illustration of the Proposition 6 for one-dimensional projections. Let H be a complex Hilbert space. Obviously, the operator $(.,f)g$ $(\neq 0)$ is a projection if and only if $(f,g) = 1$. We may assume that $\|f\| = \|g\|$.

Let $(., f)g$ be an orthogonal projection. It is evident that $(., f)g \in P_U$ if and only if vectors f,g are eigenvectors of U.

Now, let $(., f)g$ be a properly skew projection and let $(., f)g \in P_U$. Then

$$(., f)g = U^{-1}((., f)g)^*U = U^{-1}(., g)fU = (., U^{-1}g)U^{-1}f$$

Hence $f = \mu U^{-1}g$ and $g = \beta U^{-1}f$, where $|\mu| = |\beta| = 1$. Let $e^{i2\alpha} = \mu\beta$, $0 \leq \alpha < \pi$. Then $f = \mu\beta U^{-2}f = e^{i2\alpha}U^{-2}f$ and $g = e^{i2\alpha}U^{-2}g$. The projection $(.,f)g$ is a properly skew projection. This means that the numbers $e^{i\alpha}$ and $e^{i\alpha+\pi)}$ are both eigenvalues of U. Let H_α and $H_{\alpha+\pi}$ be eigensubspaces of U corresponding to $e^{i\alpha}$ and $-e^{iga}$, respectively. Denote by r_α and $r_{\alpha+\pi}$ the orthogonal projections onto H_α and $H_{\alpha+\pi}$. Let $U = \int_0^{2\pi} e^{i\lambda}de_\lambda$ be the spectral decomposition for U. We have

$$1 = (f, g) = \mu \int_0^{2\pi} e^{-i\lambda}d(e_\lambda g, g) = \mu e^{-i\alpha}((r_\alpha g, g) - (r_{\alpha+\pi}g, g))$$

Hence $\mu = \pm e^{i\alpha}$ $(=\beta)$. Similarly, $1 = \pm[(r_\alpha f,f) - (r_{\alpha+\pi}f,f)]$. Denote by S the unit sphere in H. Let $\phi \in S \cap r_\alpha H$ and $\phi^\perp \in S \cap r_{\alpha+\pi}H$ be such that $g = a\phi + b\phi^\perp$, $a, b \in R$. Then $f = \pm[a\phi - b\phi^\perp]$ and $\pm[a^2 - b^2] = 1$. Thus

$$\begin{aligned}(., f)g = &\pm(., a\phi - b\phi^1)(a\phi + b\phi^1)\\ &\pm [a^2(., \phi)\phi + ab(., \phi)\phi^1 - ab(., \phi^1)\phi - b^2(., \phi^1)\phi^1]\end{aligned}$$

where $\pm(a^2 - b^2) = 1$. For instance, if $a^2 - b^2 = 1$, then $x = e_+(., f)ge_+ = a^2(.,\phi)\phi$ and $w = (., \phi)\phi^\perp$ (see Proposition 6).

Conversely, let $e^{i\alpha}$ $(0 \leq \alpha < \pi)$ and $e^{i(\alpha+\pi)}$ be eigenvalues of U both. Let vectors ϕ, $\phi^\perp \in S$ be eigenvectors of U with respect to $e^{i\alpha}$, $-e^{i\alpha}$, respectively. Then $(., f)g \in P_U$, for $f = a\phi - b\phi^\perp$ and $g = a\phi + b\phi^\perp$, where $a, b \in R$ and $a^2 - b^2 = 1$.

Hence we have proved:

Proposition 7. (1) The logic P_U contains one-dimensional projections if and only if U has eigenvalues.

(2) The logic P_U contain one-dimensional properly skew projections if and only if there exists $\alpha \in [0, \pi)$ such that $e^{i\alpha}$ and $-e^{i\alpha}$ are both eigenvalues of U.

Note that the set of all self-adjoint, with respect to the unitary operator $e^{i\alpha}(r_a - r_{\alpha+\pi})$ $[= U/(H_\alpha \oplus H_{\alpha+\pi})]$, projections on the Hilbert space $H_\alpha \oplus H_{\alpha+\pi}$ was called a hyperbolic logic. Measures on these logics were completely described.

Corollary 8. Let $\sigma(U)$ be the spectrum of U. If $\alpha \notin \sigma(U)$ or $\alpha + \pi \notin \sigma(U)$, $\forall \alpha \in [0, \pi)$ then $P_U = \Pi_U$.

Proof. By the assumption on $\sigma(U)$, we have $\mathscr{A}_U = \{a \in B(H)\colon aU^2 = U^2a\}$. Hence, by Proposition 1, $p^*U = Up = pU$, $\forall p \in P_U$. This means that $p^* = p$, $\forall p \in P_U$.

ACKNOWLEDGMENTS

The research described in this paper was made possible in part by the Russian Foundation for Fundamental Research, grant 96-01-01265.

REFERENCES

Matvejchuk, M. S. (1989). Measure Structure on Logic of Skew Projections. I, *Izvestija Vysshikh Uchebnukh Zavedenii. Mathematica,* **8**, 39–44 [English translation, *Russian Mathematics* (*Izvestiya VUZ*), **33**(8)]

Matvejchuk, M. S. (1995). Vitaly–Hahn–Saks theorem for hyperbolic logics, *International Journal of Theoretical Physics,* **34**, 1567–1574.

Matvejchuk, M. S. (1998). Probability measures in *WJ*-algebras in Hilbert space with conjugation, *Proceedings of the American Mathematical Society,* **126**, 1155–1164.